THE ACTUAL MASS OF POTENTIAL ENERGY, II*

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1. *Problems Arising at the Junction of Classical and Relativistic Mechanics.*—In a preceding paper published under a similar title,[1] the question of the mass corresponding to potential energy was discussed in some detail. This was a typical example of the difficulties arising at the junction of two different theoretical models. Such problems were examined in a general way in a recent book by the author,[2] and the one we are discussing here is of special interest, since some of its peculiar characters seem to have been overlooked by the founders of Relativity.

The junction between Relativity and classical mechanics can be considered from two different viewpoints:

(*a*) It was generally taken for granted that *relativistic mechanics* should reduce to *classical mechanics* when the *velocity* c of light could be *made infinite.* This may be mathematically correct for relativistic mechanics of particles, but this kind of reasoning is *physically unsound.* We may make c infinite in mechanics but we cannot assume anything similar in electromagnetism. The physicist, whether he is an experimenter or a theoretician, *cannot modify the velocity of light.* This velocity c is a fundamental constant in physics. When we speak of "mechanics" in this section, it should be well specified that we are thinking only of "systems of particles"; we include problems of atoms and molecules but no continuous medium with wave propagation.

(*b*) What a physicist can do is to investigate the properties of mechanical systems of particles when dimensions and velocities remain small [L.B.I., eqs. (4) and (5)]. In such systems, the delays for the propagation of signals may be so small that they become negligible, even with the finite-light velocity c.

Conditions (*a*) and (*b*) actually lead to very different consequences. Let us, for instance, consider the mass-energy relation (E given)

$$E = Mc^2 \qquad M = \frac{E}{c^2}\cdot \tag{1}$$

In *problem a*, the mass M goes to the limit zero when c is infinite. In *problem b*, the mass M remains finite.

Conditions (*a*) may satisfy a mathematician if he is interested only in mechanics of particles, but a physicist cannot accept them under any circumstances.

Conditions (*b*) have a real physical meaning, and exhibit another serious advantage. They are consistent with low frequencies ν, hence very small quanta $h\nu$; when the energy E of the system is large in comparison to $h\nu$, we really obtain classical mechanics, where neither quanta nor relativity can play any serious role.

The definition of *potential energy* in classical mechanics is based upon the assumption that delays remain negligible for the propagation of any signals. Such an assumption is consistent with either (*a*) or (*b*) conditions. In relativistic mechanics, delays may become large, and the original definition is inapplicable. This difficulty was overcome [L.B.I., eqs. (4) and (5)] when it was proved that this type of energy should no more be considered as "potential," but became very much real, and could easily be recognized in the field of interacting particles. The proof was given for electric fields, but it can obviously be extended to most other fields.

A few more explanations are needed in order to clarify the physical meaning of the discussion given in L.B.I.

2. *Ambiguity in Special Relativity.*—The duality encountered in conditions (*a*) and (*b*) of the preceding section is much deeper than appears at first sight. We actually have to deal with *two different brands of special relativism:*

(a) *Special relativity applied to systems of particles* (where the *mass-energy* relation (1) is *used only for kinetic energy*, while potential energy obtains no mass at all): In the applications of this (*a*) theory, most authors use "given scalar and vector potentials V and $\vec{A}$" without specifying how these potentials have come into being. We have discussed these problems in some detail in L.B.I.

(b) *Special relativity in electromagnetism:* Here we have a much more comprehensive treatment, very carefully specified by Einstein and others. The mass-energy relation (1) applies for any kind of energy and all equations are consistent with a finite value of c. We proved in L.B.I. that the so-called "potential energy" of mechanics can be discovered in the energy of the electric field distributed in all space, around electric charges, and this is enough to prove that it must be given a mass.

3. *Some Examples of Practical Applications.*—In order to be able to define a potential energy, we select a technical device maintaining a static potential distribution. This is the case with a van de Graaf generator. Let us assume electrons to be emitted without velocity by a cathode at $x = 0$ where the potential is $-V$, a negative quantity; these electrons have a potential energy:

$$E_t = U_0 = -eV > 0 \quad \text{total initial energy,} \tag{2}$$

and the case of Figure 1 corresponds to the assumption

$$U_0 = +20\ m_0c^2 \approx 10 \text{ million electron-volts,} \tag{3}$$

m_0 being the rest mass of an electron, in vacuum, *at a zero potential.* According to the discussion given in L.B.I. (sections 5 and 6), we should assume that one electron at zero velocity and potential energy U_0 should obtain a total mass

$$M_0 = m_0 + {}^1/_2\ \frac{U_0}{c^2} = 11\ m_0. \tag{4}$$

The other half ${}^1/_2\ \dfrac{U_0}{c^2}$ is on the apparatus where it cannot be observed; but, for the electron itself, this is far from representing a small correction!

On the sketch of Figure 1, the energy U is supposed to decrease linearly when the distance x increases; it then reaches:

$$U = 0 \text{ at } x_0$$
$$U = -2m_0c^2 \text{ at } x_{-2}.$$

This means, according to equation (4), a rest mass

$$M = \begin{cases} m_0 \text{ at } x_0 \\ 0 \text{ at } x_{-2} \end{cases}.$$

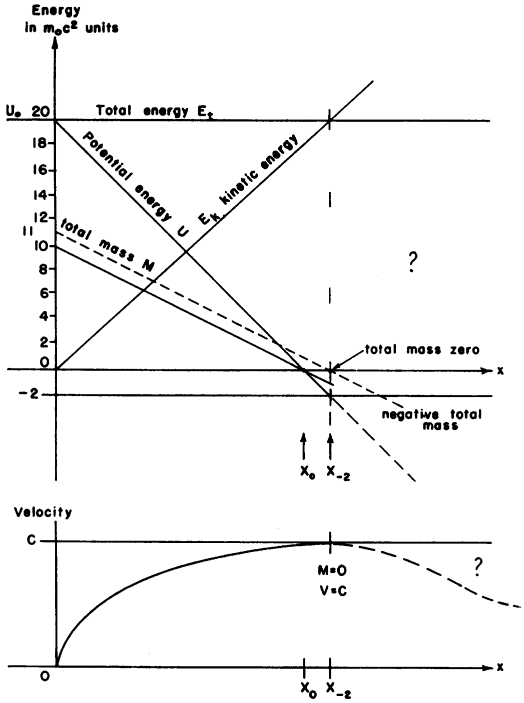

FIG. 1.

Let us assume the generator to be open in free space at x_0; electrons will be ejected with a velocity v such as

$$\frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \text{total initial energy} = U_0. \tag{5}$$

Our theory, up to now, yields the same (obvious) result as the former theories. The only change introduced is represented by equation (4) that states that the total initial mass of the electrons, at the cathode, is *eleven times* larger than usual.

Let us now assume that our system does not open into free space at x_0, but goes on beyond this stage and reaches into *negative potential energies U:*

$$\left.\begin{array}{lll} \text{at } x_{-2} & U = -2m_0c^2 & \text{total mass } M = 0 \\ \text{beyond } x_{-2} & U < -2m_0c^2 & \text{total mass } M < 0 \end{array}\right\}. \tag{6}$$

At x_{-2} the velocity v of the electron must be just equal to c, but what may happen beyond x_{-2}? We have a negative total mass

$$M = m_0 + {}^1/_2 \frac{U}{c^2} < 0, \tag{7}$$

and the kinetic energy E_k is larger than the total energy E_t

$$E_k = \frac{Mc^2}{-\sqrt{1 - \frac{v^2}{c^2}}} = E_t - U > E_t; \tag{8}$$

this requires a minus sign before the radical, to compensate for the negative total mass M!

Theoretically, there is no other alternative. The rest mass M is becoming negative, while the force remains positive, and this yields deceleration instead of acceleration. The velocity v is always smaller than c. The result may look surprising; as a matter of fact, it reminds us of the old Klein paradox in quantum theory. We proved in L.B.I. that our corrections would be small when velocities and potential energies remained small. In the present example this is no more the case, and large negative energy leads to negative mass.

At any rate, the mathematical solution corresponding to negative masses ($U < -2m_0c^2$) seems difficult to use for physical purposes, and may represent a dangerous extrapolation; this is why a question mark was added on the figure.

We must investigate the physical meaning of this mathematical discussion, and discuss the limits of applicability of the special relativity.

First, in order to avoid confusion, let us compare the present discussion with another problem, known as "hyperbolic motion." Hyperbolic motion represents the motion of a particle under a constant force (measured in a frame of reference attached to the accelerated particle!). No word is said about potential energy; the motion proceeds faster and faster and reaches asymptotically the velocity c. This discussion can be used for a linear accelerator where the field is carried by a progressing wave, whose velocity is automatically controlled and constantly matches the particles velocity. There is no potential energy to speak of in the wave and the old hyperbolic motion should apply.

Coming back to the problem sketched on Figure 1, we shall assume the particle accelerator to open in free space at x_0 since this is the definition of zero potential. In order to obtain a potential energy $-2m_0c^2$ we must have an antiparticle with opposite electric charge, and a collision must happen. Let Q' be the electric charge of the antiparticle; this means that this particle is surrounded by an electrostatic potential

$$V' = \frac{Q'}{r}, \tag{9}$$

and an approaching particle $Q = -Q'$ obtains a potential energy

$$U = +\frac{QQ'}{r} = -\frac{Q^2}{r}.$$

This potential energy between both particle and antiparticle reaches the critical value $-2m_0c^2$ at the distance r_c

$$-U_c = \frac{Q^2}{r_c} = 2m_0c^2 \tag{10}$$

that corresponds to a collision followed by annihilation.

The critical energy (10) is large enough to make up for the mass of two particles of mass m_0 each. This, however, is a process that can hardly be assumed to be correctly discussed with Relativity alone. We need quantum theory, spin, isospin etc.; annihilation of particle plus antiparticle is beyond the reach of simple Relativity.

* Contract Nonr 266 (56).

[1] Brillouin, L., "The actual mass of potential energy," these PROCEEDINGS, **53**, 475 (1965); this paper will be referred to under the initials L.B.I.

[2] Brillouin, L., *Scientific Uncertainty, and Information* (New York: Academic Press, 1964), chaps. 3, 4, and 5, and especially p. 44.

ON THE NATURE OF THE SPECTRUM OF THE TIME CORRELATION FUNCTION MATRIX FOR STOCHASTIC PROCESSES*

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If a stochastic process has a finite Poincaré recursion time,[1] the time correlation function matrix $\boldsymbol{\alpha}(t)\boldsymbol{\alpha}(t + \tau)$ is an almost periodic function of τ. Here, $\boldsymbol{\alpha}$ is a generalized vector whose components represent the departure of the observable macroscopic properties of the system from their equilibrium or steady-state values. The almost periodic nature of $\boldsymbol{\alpha}(t)\boldsymbol{\alpha}(t + \tau)$ is commonly taken to imply that the ensemble average of the time correlation function matrix $\mathbf{R}(t,\tau) \equiv \overline{\boldsymbol{\alpha}(t)\boldsymbol{\alpha}(t + \tau)}$ is also almost periodic and thus has a discrete spectrum.[2] This leads to statements implying that in order to obtain irreversible behavior and an absolutely integrable spectrum, the recurrence time must be made infinite by taking the limit of an infinitely large system. In this paper it is found that $\mathbf{R}(t,\tau)$ vanishes irreversibly as $\tau \rightarrow \infty$, for all but a small class of very special (unphysical) stochastic models. The existence of finite recursion times and the exact use of deterministic mechanics do not change this property. The argument makes explicit an opinion of Uhlenbeck,[3] and hopefully puts to rest the emphasis on the importance of Poincaré cycles in statistical mechanics.

For this study it is convenient to define certain probability distribution functions:[4] $\rho(\boldsymbol{\alpha},t)d\boldsymbol{\alpha}$ is the probability that the system be found at time t with its macroscopic state between $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha} + d\boldsymbol{\alpha}$; $\rho(\boldsymbol{\alpha},t|\boldsymbol{\alpha}', t + \tau)d\boldsymbol{\alpha}d\boldsymbol{\alpha}'$ is the joint probability that the system be found at time t in $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha} + d\boldsymbol{\alpha}$ and at time $t + \tau$ in $\boldsymbol{\alpha}'$ to $\boldsymbol{\alpha}' + d\boldsymbol{\alpha}'$; $P(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t + \tau)d\boldsymbol{\alpha}'$ is the probability the system will be found between $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}' + d\boldsymbol{\alpha}'$ at time $t + \tau$ under the condition that it be in $\boldsymbol{\alpha}$ at time t. Then,

$$\mathbf{R}(t,\tau) \equiv \overline{\boldsymbol{\alpha}(t)\boldsymbol{\alpha}(t + \tau)}$$

$$= \int d\boldsymbol{\alpha}d\boldsymbol{\alpha}'\boldsymbol{\alpha}\boldsymbol{\alpha}'\rho(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t + \tau) \quad (1)$$

$$= \int d\boldsymbol{\alpha}d\boldsymbol{\alpha}'\boldsymbol{\alpha}\boldsymbol{\alpha}'\rho(\boldsymbol{\alpha},t)P(\boldsymbol{\alpha},t|\boldsymbol{\alpha}',t + \tau). \quad (2)$$

It is customary to express $\mathbf{R}$ as a Fourier integral, based on an extension of theorems of Stone and Bochner,[5] often called the Wiener-Khinchin theorem:[6]